Circle Coordinate Geometry 1

1.

a Substitute $x = 4$, $y = 8$ into $(x - 2)^2 + (y - 5)^2 = 13$

$(x - 2)^2 + (y - 5)^2 = (4 - 2)^2 + (8 - 5)^2 = 2^2 + 3^2 = 4 + 9 = 13$ ✓

So the circle passes through (4, 8).

b Substitute $x = 0$, $y = -2$ into $(x + 7)^2 + (y - 2)^2 = 65$

$(x + 7)^2 + (y - 2)^2 = (0 + 7)^2 + (-2 - 2)^2 = 7^2 + (-4)^2 = 49 + 16 = 65$ ✓

So the circle passes through (0, –2).

c Substitute $x = 7$, $y = -24$ into $x^2 + y^2 = 25^2$

$x^2 + y^2 = 7^2 + (-24)^2 = 49 + 576 = 625 = 25^2$ ✓

So the circle passes through (7, –24).

d Substitute $x = 6a$, $y = -3a$ into $(x - 2a)^2 + (y + 5a)^2 = 20a^2$

$(x - 2a)^2 + (y + 5a)^2 = (6a - 2a)^2 + (-3a + 5a)^2 = (4a)^2 + (2a)^2 = 16a^2 + 4a^2 = 20a^2$ ✓

So the circle passes through $(6a, -3a)$.

e Substitute $x = \sqrt{5}, y = -\sqrt{5}$ into $\left(x - 3\sqrt{5}\right)^2 + \left(y - \sqrt{5}\right)^2 = \left(2\sqrt{10}\right)^2$

$$\left(x - 3\sqrt{5}\right)^2 + \left(y - \sqrt{5}\right)^2 = \left(\sqrt{5} - 3\sqrt{5}\right)^2 + \left(-\sqrt{5} - \sqrt{5}\right)^2 = \left(-2\sqrt{5}\right)^2 + \left(-2\sqrt{5}\right)^2$$

$$= 4 \times 5 + 4 \times 5 = 20 + 20 = 40 = \left(\sqrt{40}\right)^2$$

Now $\sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10} = 2\sqrt{10}$ ✓

So the circle passes through $\left(\sqrt{5}, -\sqrt{5}\right)$.

2.

The radius of the circle is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-8)^2 + ((-2)-1)^2}$$

$$= \sqrt{4^2 + 3^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

The centre of the circle is (8, 1) and the radius is 5.

So $(x - 8)^2 + (y - 1)^2 = 5^2$

or $(x - 8)^2 + (y - 1)^2 = 25$

3.

$P\,(5, 6)$ and $Q\,(-2, 2)$

The centre of the circle is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)=\left(\frac{5+(-2)}{2}, \frac{6+2}{2}\right)=\left(\frac{3}{2}, \frac{8}{2}\right)=\left(\frac{3}{2}, 4\right)$$

The radius of the circle is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(5-\frac{3}{2}\right)^2+(6-4)^2}\\ &= \sqrt{\left(\frac{7}{2}\right)^2+(2)^2}\\ &= \sqrt{\frac{49}{4}+4}\\ &= \sqrt{\frac{49}{4}+\frac{16}{4}}\\ &= \sqrt{\frac{65}{4}}\end{aligned}$$

So the equation of the circle is

$$\left(x-\frac{3}{2}\right)^2+(y-4)^2=\left(\sqrt{\frac{65}{4}}\right)^2 \text{ or } \left(x-\frac{3}{2}\right)^2+(y-4)^2=\frac{65}{4}$$

4.

Substitute $x=1,\ y=-3$ into $(x-3)^2+(y+4)^2=r^2$

$$\begin{aligned}(1-3)^2+(-3+4)^2 &= r^2\\ (-2)^2+(1)^2 &= r^2\\ 5 &= r^2\\ \text{So}\quad r &= \sqrt{5}\end{aligned}$$

5.

a $x^2 + y^2 - 10x + 4y - 20 = 0$

$x^2 - 10x + y^2 + 4y - 20 = 0$

Completing the square gives

$(x-5)^2 - 25 + (y+2)^2 - 4 - 20 = 0$

$(x-5)^2 + (y+2)^2 = 49$

b Centre of the circle = $(5, -2)$, radius = 7

6.

a $x^2 + y^2 - 2x + 8y - 8 = 0$
$x^2 - 2x + y^2 + 8y - 8 = 0$
Completing the square gives
$(x - 1)^2 - 1 + (y + 4)^2 - 16 - 8 = 0$
$$(x - 1)^2 + (y + 4)^2 = 25$$
Centre of the circle = (1, −4), radius = 5

b $x^2 + y^2 + 12x - 4y = 9$
$x^2 + 12x + y^2 - 4y = 9$
Completing the square gives
$(x + 6)^2 - 36 + (y - 2)^2 - 4 = 9$
$$(x + 6)^2 + (y - 2)^2 = 49$$
Centre of the circle = (−6, 2), radius = 7

c $x^2 + y^2 - 6y = 22x - 40$
$x^2 - 22x + y^2 - 6y = -40$
Completing the square gives
$(x - 11)^2 - 121 + (y - 3)^2 - 9 = -40$
$$(x - 11)^2 + (y - 3)^2 = 90$$
Centre of the circle = (11, 3), radius = $\sqrt{90} = 3\sqrt{10}$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$
$x^2 + 5x + y^2 - 3y = 4$
Completing the square gives
$$\left(x+\frac{5}{2}\right)^2 - \frac{25}{4} + \left(y-\frac{3}{2}\right)^2 - \frac{9}{4} = 4$$

d $\left(x+\frac{5}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{25}{2}$

Centre of the circle = $\left(-\frac{5}{2}, \frac{3}{2}\right)$, radius = $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$
$2x^2 - 8x + 2y^2 + 8y = -3$
Completing the square gives
$2((x - 2)^2 - 4) + 2((y + 2)^2 - 4) = -3$
$$2(x - 2)^2 - 8 + 2(y + 2)^2 - 8 = -3$$
$$2(x - 2)^2 + 2(y + 2)^2 = 13$$
$$(x - 2)^2 + (y + 2)^2 = \frac{13}{2}$$
Centre of the circle = (2, −2), radius = $\sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}$

7.

a $x^2 + y^2 + 12x + 2y = k$

$x^2 + 12x + y^2 + 2y = k$

$(x + 6)^2 - 36 + (y + 1)^2 - 1 = k$

$(x + 6)^2 + (y + 1)^2 = k + 37$

Centre of the circle = $(-6, -1)$

b A circle must have a positive radius, so $k + 37 > 0$

So $k > -37$

8.

$(x - k)^2 + y^2 = 41$, $(3, 4)$

Substitute $x = 3$ and $y = 4$ into the equation $(x - k)^2 + y^2 = 41$

$(3 - k)^2 + 4^2 = 41$

$k^2 - 6k + 9 + 16 = 41$

$k^2 - 6k - 16 = 0$

$(k + 2)(k - 8) = 0$

$k = -2$ or $k = 8$

9.

Substitute $y = x + 4$ into $(x-3)^2 + (y-5)^2 = 34$

$(x-3)^2 + ((x+4)-5)^2 = 34$

$(x-3)^2 + (x+4-5)^2 = 34$

$(x-3)^2 + (x-1)^2 = 34$

$x^2 - 6x + 9 + x^2 - 2x + 1 = 34$

$2x^2 - 8x + 10 = 34$

$2x^2 - 8x - 24 = 0$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

So $x = 6$ and $x = -2$

Substitute $x = 6$ into $y = x + 4$

$y = 6 + 4$

$y = 10$

Substitute $x = -2$ into $y = x + 4$

$y = -2 + 4$

$y = 2$

The coordinates of A and B are $(6, 10)$ and $(-2, 2)$.

10.

$x^2 - 4x + y^2 = 21$

Completing the square gives $(x-2)^2 + y^2 = 25$

Substitute $y = x - 10$ into $(x - 2)^2 + y^2 = 25$

$$(x-2)^2 + (x-10)^2 = 25$$
$$x^2 - 4x + 4 + x^2 - 20x + 100 = 25$$
$$2x^2 - 24x + 104 = 25$$
$$2x^2 - 24x + 79 = 0$$

Now $b^2 - 4ac = (-24)^2 - 4(2)(79) = 576 - 632 = -56$

As $b^2 - 4ac < 0$ then $2x^2 - 24x + 79 = 0$ has no real roots.

So the line does not meet the circle.

11.

a Rearranging $x + y = 11$

$y = 11 - x$

Substitute $y = 11 - x$ into $x^2 + (y - 3)^3 = 32$

$$x^2 + ((11-x)-3)^2 = 32$$
$$x^2 + (11-x-3)^2 = 32$$
$$x^2 + (8-x)^2 = 32$$
$$x^2 + 64 - 16x + x^2 = 32$$
$$2x^2 - 16x + 64 = 32$$
$$2x^2 - 16x + 32 = 0$$
$$x^2 - 8x + 16 = 0$$
$$(x-4)(x-4) = 0$$

The line meets the circle at $x = 4$ (only).

So the line is a tangent.

b Substitute $x = 4$ into $y = 11 - x$

$y = 11 - (4)$

$y = 11 - 4$

$y = 7$

The point of intersection is (4, 7).

12.

a Substitute $y = 2x - 2$ into $(x-2)^2 + (y-2)^2 = 20$

$$(x-2)^2 + ((2x-2)-2)^2 = 20$$
$$(x-2)^2 + (2x-4)^2 = 20$$
$$x^2 - 4x + 4 + 4x^2 - 16x + 16 = 20$$
$$5x^2 - 20x + 20 = 20$$
$$5x^2 - 20x = 0$$
$$5x(x-4) = 0$$

So $x = 0$ and $x = 4$

a Substitute $x = 0$ into $y = 2x - 2$
$y = 2(0) - 2$
$y = 0 - 2$
$y = -2$
Substitute $x = 4$ into $y = 2x - 2$
$y = 2(4) - 2$
$y = 8 - 2$
$y = 6$
So the coordinates of A and B are $(0, -2)$ and $(4, 6)$.

b The length of AB is

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(4-0)^2 + (6-(-2))^2}$$
$$= \sqrt{4^2 + (6+2)^2}$$
$$= \sqrt{4^2 + 8^2}$$
$$= \sqrt{16+64}$$
$$= \sqrt{80}$$
$$= \sqrt{4 \times 20}$$
$$= \sqrt{4} \times \sqrt{20}$$
$$= 2\sqrt{20}$$

The radius of the circle $(x-2)^2 + (y-2)^2 = 20$ is $\sqrt{20}$.
So the length of the chord AB is twice the length of the radius.
AB is a diameter of the circle.

Alternative method: substitute $x = 2$, $y = 2$ into $y = 2x - 2$
$2 = 2(2) - 2 = 4 - 2 = 2$
So the line $y = 2x - 2$ joining A and B passes through the centre $(2, 2)$ of the circle.
So AB is a diameter of the circle.

13.

a Substitute $y = kx$ into $x^2 - 10x + y^2 - 12y + 57 = 0$

$$x^2 - 10x + (kx)^2 - 12kx + 57 = 0$$
$$(1 + k^2)x^2 - (10 + 12k)x + 57 = 0$$

a For two distinct points of intersection, $b^2 - 4ac > 0$

$$(-(10 + 12k))^2 - 4(1 + k^2)(57) > 0$$
$$144k^2 + 240k + 100 - 228k^2 - 228 > 0$$
$$-84k^2 + 240k - 128 > 0$$
$$21k^2 - 60k + 32 < 0$$

b Using the formula, $k = \dfrac{60 \pm \sqrt{(-60)^2 - 4(21)(32)}}{2(21)}$

$$k = \frac{60 \pm \sqrt{912}}{42}$$

$k = 0.71$ or $k = 2.15$,
$0.71 < k < 2.15$

14.

$x^2 + 2x + y^2 = k$
Completing the square gives
$(x + 1)^2 - 1 + y^2 = k$

$$y^2 = k + 1 - (x + 1)^2$$

Using the equation of the line $y = 4x - 1$

$$y^2 = (4x - 1)^2$$

Solving the equations simultaneously gives

$$k + 1 - (x + 1)^2 = (4x - 1)^2$$
$$k + 1 - x^2 - 2x - 1 = 16x^2 - 8x + 1$$
$$17x^2 - 6x - k + 1 = 0$$

The line and the circle do not intersect so there are no solutions.
Using the discriminant: $b^2 - 4ac < 0$

$$36 - 4(17)(-k + 1) < 0$$
$$36 - 68 + 68k < 0$$
$$68k < 32$$
$$k < \frac{8}{17}$$

15.

Substitute $y = 2x + 5$ into $x^2 + kx + y^2 = 4$

$$x^2 + kx + (2x+5)^2 = 4$$

$$x^2 + kx + 4x^2 + 20x + 25 = 4$$

$$5x^2 + (20+k)x + 21 = 0$$

For one point of intersection, $b^2 - 4ac = 0$

$$(20+k)^2 - 4(5)(21) = 0$$

$$k^2 + 40k + 400 - 420 = 0$$

$$k^2 + 40k - 20 = 0$$

Using the formula, $k = \dfrac{-40 \pm \sqrt{40^2 - 4(1)(-20)}}{2(1)}$

$$= \frac{-40 \pm \sqrt{1680}}{2}$$

$$= -20 \pm \sqrt{420}$$

$$= -20 \pm 2\sqrt{105}$$

$k = -20 + 2\sqrt{105}$ or $k = -20 - 2\sqrt{105}$